Model Theory – Lecture 8 – Applications to the theory algebraically closed fields

In particular, we will prove

theorem ACF eliminates quantifiers

theorem ACF_p is complete
ACF_0

theorem Hilbert's Nullstellensatz

theorem (Ax) Every polinomial function $p\ \mathbb{C}^n \to \mathbb{C}^n$ that is injective is also surjective

Notice this last result is important, for example, because it started the theory of minimality and the theory of o-minimality

Exercise let $\mathcal{L}$ be a finite language and T a theory on $\mathcal{L}$

- (♡). if, given $\mathcal{M}, \mathcal{N}$ ω-saturated models, the set of finite partial isomorphism between them has the B&F, then T has QE
- (☺). if, given $\mathcal{M}, \mathcal{N}$ ω-saturated models, there is a family of partial isomorphisms with B&F, then T is complete

<u>Notation</u> • the theory of ACF is the extension of the theory of fields where we add axioms of the form

$\exists x \; p(x)=0 \qquad$ for all $p(x)\in\mathbb{Z}[X]$

- Analogously, ACF_p is $ACF \cup \{\underbrace{1+1+ \;\; +1}_{p\text{-times}} =0\}$, with $\mathbb{N} \ni p$ prime
- ACF_0 is $ACF \cup \{\neg(1+1+\ldots+1=0) \mid$ for all p prime$\}$

<u>Remark</u> • $\mathbb{Q}$ is <u>not</u> a model of ACF, ACF_p, ACF_0,

- $\mathbb{C}$ is a model of ACF and ACF_0, so is $\mathbb{C}[X]$
- $\overline{\mathbb{F}_p}$ (algebraic closure of $\mathbb{F}_p$) is a model of ACF_p

<u>Theorem</u> ACF eliminates quantifiers

<u>Proof</u> We want to use (⇐?) <u>Claim</u> we can extend the following

partial isomorphism $\mathcal{M} \xleftarrow{\omega\text{-sat}} \mathcal{N}$ to an $a \in |\mathcal{M}|$

$(a_i) \longrightarrow (b_i)$

Notice that the characteristic of the fields is the same, say p

So, the assumption tells us $\mathbb{F}_p(a_i) \simeq \mathbb{F}_p(b_i)$.

$$\begin{matrix} \mathbb{F}_p(a_i) & \simeq & \mathbb{F}_p(b_i) \\ \cap & & \cap \\ \mathcal{M} & & \mathcal{N} \end{matrix}$$

There are two cases:

-) a is algebraic over $\mathbb{F}_p((a_i))$; we consider $\mathbb{F}_p((a_i))[X]$

Let $q(x)$ be the minimal polinomial of a. Notice

$$\mathbb{F}_p((a_i))[X] \cong \mathbb{F}_p((b_i))[X]$$

and let φ be that isomorphism. Then,

$$\mathbb{F}_p((a_i))[X]/_{q(x)} \simeq \mathbb{F}_p((b_i))[X]/_{\varphi(q(x))}$$

$$\begin{matrix} \| \wr \\ \mathbb{F}_p((\overline{a_i})) \subset \mathcal{M} \end{matrix}$$

↳ this is the tuple (a_i) with a concatenated

(*) Notice that b exists since $\mathcal{N}$ is ω-saturated

Since $\mathcal{N}$ is ω-saturated, $\varphi(q(x))$ has a zero in $\mathcal{N}$, call it b

So $\mathbb{F}_p((\overline{b_i})) \cong \mathbb{F}_p((b_i))[X]/_{\varphi(q(x))}$, and we can extract the

new partial isomorphism

-) a is transcendental. Then, $\mathbb{F}_p((a_i))(a) \cong \mathbb{F}_p((a_i))[X] \simeq \mathbb{F}_p((b_i))[X]$,

so we choose a transcendental $b^{(*)}$ over $\mathbb{F}_p((b_i))$ and link a to it

Theorem ACF_0 is complete (Same for ACF_p)

Proof idea We will prove a formula is true in a model iff it is true in every model

Proof Take φ, eliminate the quantifiers and get $\overline{\varphi}$ (previous theorem). Notice $\overline{\varphi}$ is just "numbers" and we check it in the base field ∎

Proposition If a theory Φ eliminates quantifiers and it has a model that embeds in every other model, then Φ is complete

Proof Same argument as above

(Lefschetz principle)

Corollary Let σ be a formula in $\mathcal{L}_{\mathbb{F}}$ (language of fields). The following are equivalent

1) $\mathbb{C} \models \sigma$,
2) $\mathbb{F} \models \sigma$ for all $\mathbb{F}$ algebraically closed and of characteristic 0,
3) For an infinite number of primes p, σ is true in all algebraically closed fields of characteristic p (J_σ)

Proof ($1 \Leftrightarrow 2$) Follows from completeness

($1 \Rightarrow 3$) If σ is true, then by compactness we only use a finite number of $\{\neg(\underbrace{1+1+\dots+1}_{p}=0) \mid p \text{ prime}\}$ The complement works

($3 \Rightarrow 1$) Assume $\mathbb{C} \not\models \sigma$ Then $\mathbb{C} \models \neg\sigma$ This produces a co-finite set of primes I, and must intersect J ↯ ∎

<u>Theorem (Ax)</u> Let $f\ \mathbb{C}^n \to \mathbb{C}^n$ polynomial and injective f is surjective

<u>Proof</u> Let σ be $\forall y_1, \dots, y_n \in \mathbb{C}\ (\exists z_1, \dots, z_n \in \mathbb{C}\ (\bigwedge_{1 \le i \le n} f_i(z_1, \dots, z_n) = y_i))$ (*)

Notice this applies to $f_{p^e}\ \mathbb{F}_{p^e}^n \to \mathbb{F}_{p^e}^n$ as well, with $f_{p^e} \equiv f\ (p^e)$

Then, the statement is trivial as this is finite stuff. Then,

$$\bigcup_{e \in \mathbb{N}} \mathbb{F}_{p^e} \xrightarrow{f^\infty} \bigcup_{e \in \mathbb{N}} \mathbb{F}_{p^e},$$

where, of course $\overline{\mathbb{F}_p} = \bigcup_{e \in \mathbb{N}} \mathbb{F}_{p^e}$, and f^∞ is the union of the f_{p^e}

This concludes, because, for every p prime, σ is true in $\overline{\mathbb{F}_p}$,

so the Lefschetz principle $(3 \Rightarrow 1)$ gives the main result

<u>Proposition</u>. (Weak Nullstellensatz) Let $\mathbb{K}$ be algebraically closed. Let

$p_i(x_1, \dots, x_n) = 0$, for $1 \le i \le h$, for some h finite, be polynomial

equations. If this has a solution in L (extending $\mathbb{K}$), then it

has a solution in $\mathbb{K}$

<u>Proof</u> We can assume L is algebraically closed (we can close it). Then,

by model completeness we concluded ∎

<u>Theorem</u> (Nullstellensatz) If f is a polynomial on $\mathbb{K}$ algebraically closed

and f_i is a finite family of polynomials such that

$$\forall i \; f_i(x) = 0 \longrightarrow f(x) = 0,$$

then there exists a number l such that $f^l \in (\langle f_i \mid i \rangle)$ in $\mathbb{K}[x]$

<u>Proof</u>: (Rabinowitsch trick) By the assumption, we know that

$\{f_1, f_2, \dots, f_n, 1 - yf\}$ have no common zero in $\mathbb{K}$. Since

$\mathbb{K}$ is algebraically closed, they have no common zero in any extension

of $\mathbb{K}$ by weak-nullstellensatz. Then $\langle f_1, \dots, f_n, 1 - yf \rangle$ must be

the same as $\langle 1 \rangle$ in $\mathbb{K}[x][y]$, otherwise we extend it to maximal

and $\mathbb{K}[x][y] / _{M_{\langle f_1, \dots f_n, 1-yf \rangle}}$ is a field extending $\mathbb{K}$ that has such a root we

proved wasn't there. Then, $\sum a_i f_i^{x_i} + a(1-yf)^e = b$ (invertible)

The argument concludes since $\mathbb{K}[x](y)$ is a field and $y = \frac{1}{f} \cdot 1$ ... ∎